Minimal Numbers of Fox Colors and Quandle Cocycle Invariants of Knots

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Abstract

Relations will be described between the quandle cocycle invariant and the minimal number of colors used for non-trivial Fox colorings of knots and links. In particular, a lower bound for the minimal number is given in terms of the quandle cocycle invariant.

1 Introduction

The determinant of a knot is divisible by a prime p if and only if a diagram of the given knot is (Fox) p-colorable [5]. Since the knot determinant often takes much larger values than the crossing numbers for prime alternating knots with prime determinants, it was conjectured [6] that for any prime p, if an alternating knot K has the determinant p, then any non-trivial coloring of any minimal crossing diagram of K assigns distinct colors on its arcs (*Kauffman-Harary conjecture*). The conjecture stays open at the time of writing after extensive studies of wide variety of families of knots (see, for example, [1]). Considering this situation the following example is interesting, that was discovered by I. Teneva (described in [6]): the $(2, 5)$-torus knot $T(2, 5)$, which has determinant 5, has a non-alternating, non-minimal diagram with only 4 colors, as depicted in Fig. 1(B) (compare with its minimal alternating diagram colored by 5 distinct colors as shown in Fig. 1(A).) In [8], the minimum number of p-colors $\mathrm{mincol}_p(K)$ for knots K was further studied, and it was proved that if $\mathrm{mincol}_p(K) = 3$, then 3 divides the determinant of a knot K. From these facts, the case $\mathrm{mincol}_p(K) = 4$ is of interest, and is a focus of this paper.

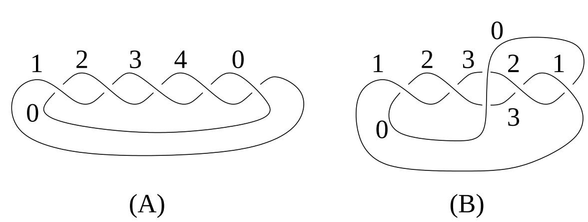

Figure 1: Teneva's example

Quandle cocycle invariants, defined using quandle cohomology theory, have been studied and applied to knots and knotted surfaces [2, 3] in the past several years. For Fox p-colorings with region colors for knots and links L, quandle cocycle invariant $\Phi_p(L)$ is defined and written as a multiset (a set with repetitive elements allowed) in $\mathbb{Z}_p$ (more details will be given in Section 2).

For example, monochromatic (trivial) colorings of a knot contribute copies of $0 \in \mathbb{Z}_p$ in $\Phi_p(L)$, but some non-trivial colorings may further contribute 0 as well. Hence if $\Phi^0_p(L)$ denotes the number of zeros in $\Phi_p(L)$, then $\Phi^0_p(L) \geq p^2$ for any link L, and $\Phi^0_p(L) > p^2$ if and only if there is a non-trivial p-coloring that contributes 0 to the invariant $\Phi_p(L)$. A link is called *non-split* if there is no embedded 2-sphere in 3-space that separates the link non-trivially. In this paper we prove

Theorem 1.1 (1) *If there exists a prime $p > 7$ such that a non-split link L satisfies $\Phi^0_p(L) = p^2$, then* $\mathrm{mincol}_q(L) \geq 5$ *for every prime $q > 7$.*
(2) *There exist infinitely many prime alternating knots K with $\Phi^0_7(K) = 49$, and their diagrams that are 7-colored with exactly 4 colors.*

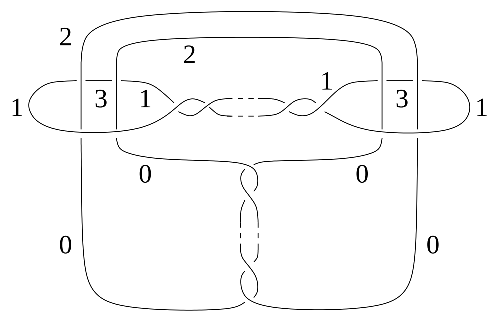

Figure 2: Links colored with 4 colors

For the statement (1), we exhibit in Fig. 2 a family of examples of non-split links that are p-colorable for any prime p, with exactly 4 colors. The smallest example of the statement (2) of the theorem is depicted in Fig. 3, which is equivalent to 5_2, the first knot in the table that is 7-colorable.

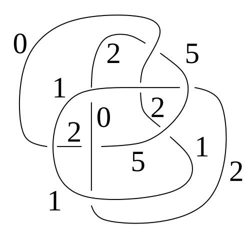

Figure 3: A 7 coloring of 5_2 with 4 colors

In Section 2, we present definitions necessary to prove the main theorem, and the proof is given in Section 3.

2 Preliminaries

Fox p-colorings are well known in knot theory, and their descriptions can be found in [3, 7, 12], for example. The definition of quandles, as well as references are also found in [3, 7]. In this section we give brief reviews needed for the statement and proof of Theorem 1.1, and leave details to these references. The set $\mathbb{Z}_p$ with a binary operation $(x, y) \mapsto x * y = 2y - x \mod p$ for $x, y \in \mathbb{Z}_p$ is

called the *dihedral quandle* of order p. A map from the set of arcs by $\mathbb{Z}_p$ is called a Fox p-coloring, if it satisfies the condition in Fig. 4 for y, z assigned on arcs. A coloring is regarded as assigning elements of a quandle to arcs, and the element assigned to an arc is called the *color* of the arc. A map from the set of arcs and regions of a planar knot diagram is called a Fox coloring *with region colors*, if it satisfies the condition in Fig. 4, where the region colors (elements assigned to regions) are framed by squares. In fact, for defining Fox colors, orientations of knots are not necessary.

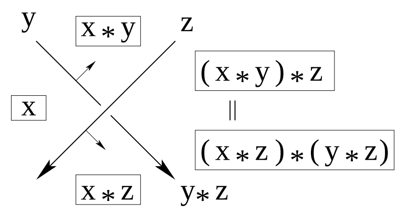

Figure 4: Coloring rules of arcs and regions

Definitions and references for quandle cocycle invariants are found in [3], for example. Although orientations are needed to define the invariant for general quandles, it is shown by Satoh [13] that it is well-defined without orientation for dihedral quandles, and we use his result for the definition below. Let $\mathcal{C}$ be a Fox p-coloring with region colors of a given diagram K. Refer to Fig. 4 for the descriptions below. Select one of the four regions near a crossing τ, and call it the *source region*. Let x_τ be the color of the source region. Select one of the two under-arcs at τ, and call it the *source under-arc*. Let y_τ be the color of the source under-arc, and z_τ be the color of the over-arc at τ, then (x_τ, y_τ, z_τ) is called the *ordered triple of colors* at a crossing τ. Let n_y and n_z be the normal vectors of the source under-arc and the over-arc at τ, respectively, such that they point from the source region across the arcs to the other regions. The local sign of τ denoted by $\epsilon(\tau)$, which depends on the choices of the source region and the source under-arc, is defined to be positive if the ordered vectors (n_z, n_y) agrees with the orientation of the plane, and negative otherwise. The weight at τ for the coloring $\mathcal{C}$ is defined by $B(\mathcal{C}, \tau) = \epsilon(\tau)\phi(x_\tau, y_\tau, z_\tau)$, where ϕ is called a *quandle* 3*-cocycle* described below. The quandle (3-)cocycle invariant is defined by the multiset $\Phi_p(K) = \{\sum_\tau B(\mathcal{C}, \tau) \mid \mathcal{C}$ ranges over all p-colorings with region colors$\}$. This is a knot invariant, which follows from the properties satisfied by ϕ, in particular, the condition called the quandle 3-cocycle condition (see [3]). For dihedral quandles, Mochizuki [9] gave the following explicit formula:

$$\phi(x, y, z) = (x - y)[(2z^p - y^p) - (2z - y)^p]/p \in \mathbb{Z}_p.$$

It was proved in [13] that the weight $B(\mathcal{C}, \tau)$ does not depend on the choices of the source region and the source under-arc when Mochizuki's cocycle is used to define the weight.

As was mentioned in Section 1, trivial colorings (those with the same color on all arcs) contribute 0 to $\Phi_p(K)$. This is seen from $\phi(x, y, z)$ by setting $y = z$. The contribution is 0 regardless of region colors, but a color of one region (say, the region at infinity) uniquely determines colors of all the other regions, so that there are at least p^2 copies of 0's in $\Phi_p(K)$ for any p and K. Let $\Phi_p^0(K)$ denote the number (multiplicity) of 0's in $\Phi_p(K)$, then we have $\Phi_p^0(K) \geq p^2$, and the inequality is strict if and only if there exists a coloring with region colors that is non-trivial on arcs, yet contributes 0.

For dihedral quandles, we take advantage of the fact that $\mathbb{Z}_p$ forms a field. Let p be a prime,

and $c \in \mathbb{Z}_p$. If $c \neq 0$, then c is invertible in the field $\mathbb{Z}_p = \mathbb{F}_p$, and denote by $(1/c)$ the multiplicative inverse of c. For a p-coloring of a knot diagram and $c \in \mathbb{Z}_p$, define a coloring $c + \mathcal{C}$ and $c\, \mathcal{C}$ to be p-colorings defined by $(c + \mathcal{C})(\alpha) = c + \mathcal{C}(\alpha)$ and $(c\ \mathcal{C})(\alpha) = c \cdot \mathcal{C}(\alpha)$, respectively, for every arc α of the diagram. It is easily seen that they are well-defined. Similarly, if $c \neq 0$, define a coloring $(1/c)\mathcal{C}$ by $((1/c)\mathcal{C})(\alpha) = (1/c) \cdot \mathcal{C}(\alpha)$ for every arc α of the diagram.

3 Proof of Theorem 1.1

Let L be a non-split link. To prove the statement (1), suppose that a diagram D of L has a p-coloring $\mathcal{C}$ for a prime $p > 7$ with 4 colors. Then we prove that the coloring $\mathcal{C}$ can be defined for any prime $q > 7$, and makes a trivial contribution $(0 \in \mathbb{Z}_p)$ to the cocycle invariant $\Phi_q(L)$, and therefore $\Phi_q(L) > q^2$ for any prime $q > 7$.

Let $\mathcal{C}(D)$ denote the set of colors that appear in $\mathcal{C}$. By replacing $\mathcal{C}$ by $(-c) + \mathcal{C}$ for a color $c \in \mathcal{C}(D)$, we may assume that $0 \in \mathcal{C}(D)$. Let $c \neq 0$ be a color that is assigned to the over-arc at the end of an arc colored by 0, so that at this crossing τ, one of the under-arcs has color 0 and the over-arc has color c. By replacing $\mathcal{C}$ by $(1/c)\mathcal{C}$, we map assume that $c = 1$. Then at τ, the other under-arc is colored by 2. The situation is depicted at the left crossing of Fig. 5.

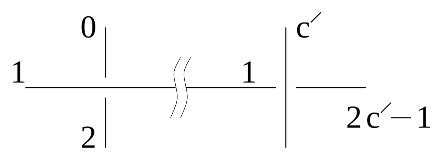

Figure 5: Specifying 4 colors

We may assume that there is an arc colored by 1 that ends at an over-arc colored with $c' \neq 1$ at a crossing τ', since otherwise, there are components colored only by 1 that lie above all the other components, and this contradicts the assumption that L is non-split. Let c' be the color of the over-arc of this crossing τ' that is an end point of an arc colored 1, so that the three colors at τ' are $1, c'$, and $2c' - 1$. Since $\mathcal{C}(D)$ consists of 4 colors and contains $\{0, 1, 2\}$, either $c' \in \{0, 1, 2\}$ (and we assumed $c' \neq 1$), or $2c' - 1 \in \{0, 1, 2\}$. If $c' = 0$, then the other under-arc has color -1, and by considering $1 + \mathcal{C}$, we may assume that $\mathcal{C}(D) = \{0, 1, 2, 3\}$. If $c' = 2$, then the other under-arc has color 3, and again we may assume that $\mathcal{C}(D) = \{0, 1, 2, 3\}$. If $2c' - 1 = 0$, then $c' = 1/2$, and $\mathcal{C}(D) = \{0, 1, 2, 1/2\}$. If $2c' - 1 = 1$ or c', then $c' = 1$, which contradicts our choice of $c' \neq 1$. If $2c' - 1 = 2$, then $c' = 3/2$, and $\mathcal{C}(D) = \{0, 1, 2, 3/2\}$.

Case 1: $\mathcal{C}(D) = \{0, 1, 2, 3\} \subset \mathbb{Z}_p$. If there is a crossing at which the over-arc is colored by 0, then the under-arcs are colored by unordered pairs of elements of $\mathbb{Z}_p$ $(0, 0)$, $(1, -1)$, $(2, -2)$ or $(3, -3)$, and only $(0, 0)$ is possible since $p > 5$. For an over-arc colored 3, possible colors of the under-arcs are $(0, 6)$, $(1, 5)$, $(2, 4)$ and $(3, 3)$, all numbers considered in $\mathbb{Z}_p$, and only $(3, 3)$ is possible since $p > 5$. Hence 0 and 3 cannot be a color of an over-arc other than at a trivially colored crossing.

If 1 is at an over-arc, then the possible colors of under-arcs are $(0, 2)$, $(1, 1)$ and $(3, -1)$, the last of which is impossible. For 2 at an over-arc, then possible colors of the under-arcs are $(0, 4)$, $(1, 3)$

and $(2,2)$, the first of which is impossible. In summary, the possible colors at a crossing are, other than a constant coloring at a crossing, the over-arc 1, under-arcs $(0,2)$ or the over-arc 2, under-arcs $(1,3)$. Note that the arc colored 0 always ends at an over-arc colored 1, and 3 ends at 2. Note also that this coloring rule by $\{0,1,2,3\}$ is valid for any p.

At this point we know that there is no knot that is colored by this pattern, since the diagram is non-trivially colored for any p, and the existence of such colorings would imply [5] that the determinant of the knot is divisible by all primes p, a contradiction (knot determinant takes values in odd integers, see [12], for example).

Now we evaluate the contribution to the cocycle invariant of this coloring with an arbitrary fixed region colors for the link L. Let $(D_0, \mathcal{C}_0)$ be this coloring by $\{0,1,2,3\}$ of the given diagram, and let $B(D_0, \mathcal{C}_0)$ be the evaluation by the Mochizuki 3-cocycle (which is the contribution to the cocycle invariant $\Phi_p(L)$ of this coloring). Let D_1 be the diagram obtained from D_0 by performing (arbitrary) smoothings at all crossings colored by constant colors $0,1,2$ or 3, and give the inherited coloring $\mathcal{C}_1$ to obtain $(D_1, \mathcal{C}_1)$. The link type of D_1 may no longer be L, but we retain $B(D_1, \mathcal{C}_1) = B(D_0, \mathcal{C}_0)$. Every arc colored 0 or 3 of the diagram D_1 has no crossing with constant (trivial) colors after smoothings, and they end at over-arcs with distinct colors. Every crossing has a single under arc colored by either 0 or 3, exclusively. Hence the set of arcs colored by 0 or 3 gives rise to pairings of crossings, by declaring that the end points of an arc colored by 0 or 3 are paired. The contribution to the weight $B(D_1, \mathcal{C}_1)$ of each pair is zero. This is seen by selecting a region shared by the pair of crossings as the source region, and the shared under-arc as the source under-arc. Then the crossings have the same ordered triple of colors and opposite local signs, see Fig. 6. Hence we have $B(D_1, \mathcal{C}_1) = B(D_0, \mathcal{C}_0) = 0$, a contradiction to the assumption that $\Phi_p(L) = p^2$. The argument is valid for all prime $q > 7$.

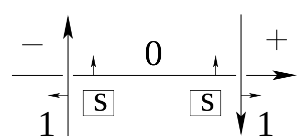

Figure 6: Pair crossings' contributions cancel

Case 2: $\mathcal{C}(D) = \{0,1,2,1/2\} \subset \mathbb{Z}_p$. We may assume that these are distinct, and that $1/2 \neq 3$. If there is a crossing at which the over-arc is colored by 0, then possibilities of colors of the under-arcs are unordered pairs $(0,0)$, $(1,-1)$, $(2,-2)$ and $(1/2,-1/2)$. All colors must be from $\mathcal{C}(D) = \{0,1,2,1/2\}$. Since $p > 5$ we have $\{-1,-2\} \neq \{1,2\}$, as well as $1/2 \neq -1/2$, $-1/2 \neq 1$, $-1 \neq 1/2$, $-2 \neq 1/2$. Hence only $(0,0)$ is possible. For an over-arc colored 1, the under-arcs are colored $(0,2)$, $(1,1)$, or $(1/2,3/2)$, but $3/2$ cannot be any of $\mathcal{C}(D) = \{0,1,2,1/2\}$, so only $(0,2)$ and $(1,1)$ are possible. If 2 is at an over-arc, then the under-arcs may be colored $(0,4)$, $(1,3)$, $(2,2)$ or $(3/2,5/2)$, and again only $(2,2)$ is possible if $p > 7$, and $(0,4)$ is possible for $p = 7$ where $\mathcal{C}(D) = \{0,1,2,1/2=4\}$. For $1/2$ at an over-arc, the under-arcs are colored $(0,1)$ or $(1/2,1/2)$. In summary, the possible colors at a crossing are, other than the constant coloring at a crossing, the over-arc 1, under-arcs $(0,2)$, and the over-arc $1/2$, under-arcs $(0,1)$. Since $1/2$ does not appear on

an under-arc other than crossings colored by a constant $1/2$, the components colored by $1/2$ lies above all the others, and contradicts that L is non-split. Hence this case does not occur for $p > 7$.

Case 3: $\mathcal{C}(D) = \{0, 1, 2, 3/2\} \subset \mathbb{Z}_p$. We may assume that these are distinct, and that $3/2 \neq 3$. If there is a crossing at which the over-arc is colored by 0, then possible colors of the under-arcs are pairs $(0, 0)$, $(1, -1)$, $(2, -2)$ and $(3/2, -3/2)$. Since $p > 5$ we have $\{-1, -2\} \neq \{1, 2\}$, $3/2 \neq -3/2$, $-3/2 \neq 0, 1$, and $p - 1 \neq 3/2$. Hence $(1, -1)$ and $(3/2, -3/2)$ are impossible. If $-2 = 3/2$, or $-3/2 = 2$, then $p = 7$. Hence $(1, -1)$ is impossible for any $p > 5$ and $(2, -2)$ and $(3/2, -3/2)$ are possible only for $p = 7$, in which case both are $(5, 2)$. For over-arc colored 1, the under-arcs are colored $(0, 2)$, $(1, 1)$ or $(3/2, 1/2)$, and $(3/2, 1/2)$ is impossible as $1/2 \neq 0, 1, 2, 3/2$ for odd prime $p > 5$. If 2 is at an over-arc, then the under-arcs are colored $(0, 4)$, $(1, 3)$, $(2, 2)$ or $(3/2, 5/2)$, and only $(2, 2)$ is possible since $p > 5$. For $3/2$ at an over-arc, the under-arcs are colored $(0, 3)$, $(1, 2)$ or $(3/2, 3/2)$, and $(0, 3)$ is impossible. In summary, the possible colors at a crossing are, other than a constant coloring at a crossing, the over-arc 0, under-arcs $(2, 5)$ if $p = 7$, the over-arc 1, under-arcs $(0, 2)$ and the over-arc $3/2$, under-arcs $(1, 2)$. Note that 2 cannot be a color of an over-arc other than at a trivially colored crossing. If $p > 7$, then $3/2$ does not appear as a color of an under-arc, and therefore the components colored by $3/2$ splits, and it is a contradiction. Hence the proof is complete for $p > 7$.

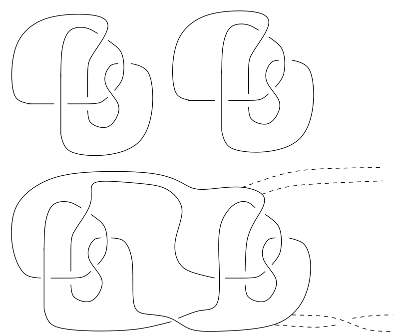

Figure 7: Connecting 5_2

For $p = 7$, an example of a knot that satisfies the condition stated in the theorem is already exhibited in Fig. 3. It remains to construct an infinite family of such. In Fig. 7, a construction is illustrated to obtain such an infinite family by connecting any finite number of copies of 5_2. From the construction, it is reduced alternating, hence they are all distinct for distinct numbers of copies used, from [10, 14]. The connection is made at arcs with the same color in Fig. 3, and by going back to copies of diagrams in Fig. 3, the colorings with only four colors are obtained. Finally, we observe that for infinitely many of them, every non-trivial coloring contributes non-trivial values to the cocycle invariant. Let K_n be the knot constructed above with n copies of 5_2. Any non-trivial coloring of 5_2 contributes 1, 2, or $4 \in \mathbb{Z}_7$ to the invariant $\Phi_7(K_n)$. Then for each contribution, the contribution of the induced coloring for K_n is n, $2n$, and $4n \in \mathbb{Z}_7$, respectively, so that the invariant is non-trivial for all n not divisible by 7. This provides an infinite family as required. □

Remark 3.1 It was proved in [11], in fact, that any 7-colorabe knot has a diagram with exactly four colors. Using her result, Theorem 1.1 (2) can be proved simply by providing infinitely many

7-colorable knots K with $\Phi_p^0(K) = 49$.

Remark 3.2 In terms of quandle homology theory [4], Theorem1.1 can be restated as follows: (1) For any $p > 7$, any p-coloring of a link diagram with 4 colors represents a null-homologous 3-cycle in $H_3^{\mathrm{Q}}(\mathbb{Z}_p, \mathbb{Z}_p)$. (2) There exist infinitely many prime alternating knots K, and 7-colorings of their diagrams with exactly 4 colors, that represent non-zero homology classes of $H_3^{\mathrm{Q}}(\mathbb{Z}_7, \mathbb{Z}_7)$. The minimal number of elements of a quandle used to represent cycles that are non-trivial in quandle homology is of interest from this view point.

Acknowledgments

The author was supported in part by NSF Grant DMS #0603876. He is thankful to J. Scott Carter, L.H. Kauffman, P. Lopes and K. Oshiro for valuable conversations and comments.

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